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1989 J. Phys.: Condens. Matter 1 915

(http://iopscience.iop.org/0953-8984/1/5/008)

J. Phys.: Condens. Matter 1 (1989) 915–928. Printed in the UK

Lack of re-entrance in certain Ising spin-glass models

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Received 10 August 1988

Abstract. We report results of numerical transfer matrix calculations on a dilute two-dimensional first- and second-neighbour Ising model and Migdal–Kadanoff renormalisation group studies on a random nearest-neighbour Ising model in two and three dimensions. The calculations were carried out to examine the existence of re-entrance in these systems, defined by the disappearance of long-range ferromagnetic order upon lowering the temperature. We find, contrary to other claims in the literature, that there is no re-entrance in these models or, at best, re-entrance occurs over a very narrow range of parameters, and speculate about the necesary ingredients to obtain re-entrance in random spin systems.

1. Introduction

In many spin-glass systems there is a range of concentration of the atomic species where, upon lowering the temperature, the successive phases are paramagnetic, ferromagnetic and spin glass (Binder and Young 1986, Maletta and Zinn 1986). Such systems are called re-entrant spin glasses. The terminology is somewhat inaccurate, since the system does not *re-enter* a spin-glass state at low temperatures, as it was a paramagnet, not a spin glass, at high temperatures, but it does, however, re-enter a *less ordered* state. This succession of phases implies that the re-entrant spin-glass phase has the lowest energy, while the ferromagnetic phase has more entropy, which is intuitively rather surprising.

Re-entrance, or reappearing phases, have been observed in a variety of systems that undergo some kind of phase transition. A very simple example is seen in binary liquid mixtures (Walker and Vause 1980, 1987), while other examples include superconductivity (Lin *et al* 1984), and liquid crystals (Indekeu and Berker 1986). The usual mechanism responsible for re-entrant behaviour is the existence of some hidden interaction that is capable of lowering the entropy of the system while reducing its energy. No such interaction, or other mechanism, has been identified in spin glasses as yet.

Re-entrance has been found in a wide variety of spin glasses: insulating as well as metallic, and both crystalline and amorphous alloys (Maletta and Zinn 1986, Aeppli *et al* 1983, 1984a, b, Wong *et al* 1985a, b, 1987). The interpretation of the experimental findings, in particular the nature of the low-temperature phase is, however, still a subject of controversy (Maletta and Zinn 1986, Coles 1984). One well studied example is the insulator $Eu_xSr_{1-x}S$, which is known (Maletta and Felsch 1979, Binder *et al* 1979, Kinzel and Binder 1981) to be a text-book example of a Heisenberg system. It can be modeled successfully by a Heisenberg magnet on an FCC lattice that has a first-neighbour ferromagnetic coupling (J_1) and antiferromagnetic second-neighbour interaction (J_2) of

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smaller magnitude, where, to a good approximation, $0 < J_1/2 = -J_2$. This system shows re-entrant ferromagnetic behaviour in the concentration range of the magnetic species $0.51 \leqslant x \leqslant 0.65$.

Most of the systems in which re-entrance has been studied are Heisenberg ferromagnets. However, there have also been measurements on the Heisenberg antiferromagnet $Eu_xSr_{1-x}Te$ (Borgermann *et al* 1987) and the Ising antiferromagnet $Fe_xMg_{1-x}Cl_2$ (Wong *et al* 1985a, b, 1987). Unfortunately though, no Ising ferromagnetic systems have been studied to our knowledge. $Eu_xSr_{1-x}Te$ is a diluted Heisenberg system with dominating antiferromagnetic interactions, for which experiments find a spin-glass regime in the concentration range $0.15 \leqslant x \leqslant 0.5$. There is no indication of re-entrance, neither from irreversibility effects, nor from the AC susceptibility results. Unfortunately, no neutron scattering experiments have been done as yet (Borgermann *et al* 1987). $Fe_{1-x}Mg_xCl$ is a diluted Ising antiferromagnet with competing first- and second-neighbour interactions. The lower transition occurs at $T \simeq 3$ K for $x \simeq 0.45$, as found by AC susceptibility measurements. Neutron diffraction experiments, however, reveal that long-range antiferromagnetic order and spin-glass-like short-range order *coexist* in this phase (Wong *et al* 1985a, b, 1987). Thus this system is not re-entrant according to our definition. To sum up the experimental findings: true re-entrance has been found in Heisenberg ferromagnetic systems only. Heisenberg antiferromagnets do not show true re-entrant behaviour, neither do Ising antiferromagnetic systems.

The experimental discovery of re-entrance did not come as a surprise, since the early mean-field theory of Ising spin-glass models (Sherrington and Kirkpatrick 1975) did show this behaviour. It was only later, when the theory of Parisi (1979, 1980a, b, c) appeared, that it became apparent that re-entrance does *not* occur (Toulouse 1980) in the correct mean field theory of spin glasses.

There have been a number of reports in the literature recently (Katsura and Matsuno 1983, Wolff and Zittartz 1985, 1986, Benayad *et al* 1987) that claim to provide either a rather simple explanation of re-entrant behaviour or claim to demonstrate that re-entrance occurs very generally in short-range disordered systems, in particular in certain simple random Ising models with short-range interactions. Motivated by these claims we have carried out careful studies of these models and find, by contrast, that they do not exhibit re-entrance. Nonetheless we believe that our results *are compatible* with experimental findings, since the one Ising system studied, $Fe_{1-x}Mg_xCl_2$, does not show re-entrance either. We suspect that one must have Heisenberg spins and/or include long-range interactions such as dipole–dipole couplings, to obtain re-entrant behaviour.

2. Theoretical models

The effect of the different concentrations of the atomic species can be mimicked in theoretical models by changing the ratio of the mean to the standard deviation of the distribution of interactions. The behaviour of such models is well understood only for the case of infinite range interactions. For these models with Ising spins, if this ratio is such that ferromagnetic ordering occurs as the temperature is lowered, then this long-range order is never lost as the temperature is further decreased. The mean-field phase diagram, obtained (Parisi 1979, 1980a, b, c) from the exact solution of the infinite range Sherrington–Kirkpatrick (1975) model, is shown in figure 1(*a*). Spin-glass behaviour does set in at low temperatures, but it coexists with the ferromagnetic long-range order. There is a 'mixed' phase with ferromagnetism and replica symmetry breaking (i.e. irreversibility) separated from the true ferromagnetic phase (replica symmetric, no

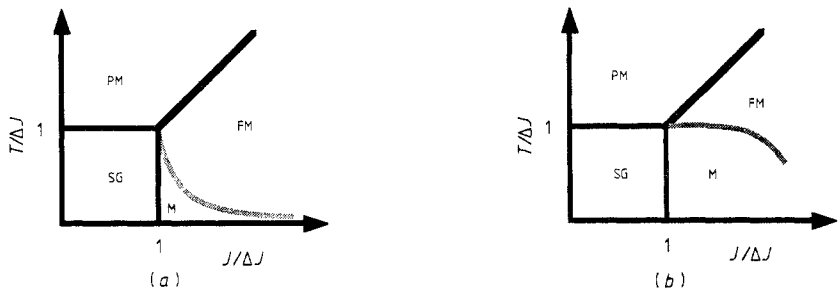

Figure 1. Infinite-range model phase diagrams: The mean-field transition temperatures are plotted against the ratio of the mean and the standard deviation of the bond distribution. The phases are: PM, paramagnetic; FM, ferromagnetic; M, mixed; and SG, spin glass. The SG–FM phase boundary is vertical. (*a*) Ising spins. The shaded curve, known as the AT line, separates the mixed phase, which has both ferromagnetic ordering and replica symmetry breaking (irreversibility), from the FM phase, which has no replica symmetry breaking. (*b*) Heisenberg spins. The shaded curve, known as the GT line, separates the mixed phase, which has ferromagnetism coexisting with transverse spin-glass order, from the purely ferromagnetic phase.

irreversibility) by the Almeida–Thouless (AT) line (de Almeida and Thouless 1978). Ironically, the earlier Sherrington–Kirkpatrick (SK) theory (Sherrington and Kirkpatrick 1975) did show re-entrance. However, the replica symmetric solution turned out to be unstable below the AT line, and the presumably exact Parisi (1979, 1980a, b, c) solution gives a vertical spin-glass–ferromagnet (SG–FM) phase boundary (Toulouse 1980). Thus we see that the results of the measurements on the antiferromagnetic Ising system $Fe_{1-x}Mg_xCl_2$ are in qualitative agreement with the predictions of the mean-field theory.

For Heisenberg spins the situation in mean-field theory is very similar, except that *transverse* spin-glass order also sets in at lower temperatures (Gabay and Toulouse 1981). The phase diagram is shown in figure 1(*b*). There is again a mixed phase, where ferromagnetic long-range order coexists in this case with transverse spin-glass order. This is separated from the true ferromagnetic phase by the Gabay–Toulouse (GT) line below which replica symmetry breaking occurs. But the SG–FM phase boundary is vertical, just as in the Ising case. Thus there is no re-entrance in non-diluted infinite range models of spin glasses.

Recently a generalised *dilute* infinite range Ising spin-glass model has been introduced and studied (Viana and Bray 1985). The model is used to represent $Eu_xSr_{1-x}S$, by taking ferromagnetic bonds with probability 2/3 and antiferromagnetic ones with probability 1/3 and half strength. Apart from the paramagnetic (PM), ferromagnetic (FM) and spin-glass (SG) phases, a mixed phase is found, but the phase boundary between the SG and FM phases can be *re-entrant*, depending upon the parameters of the model.

Short-range models of spin glasses are less well studied and understood. For simplicity most of the studies of re-entrance have been done on Ising models. Results are available in the annealed approximation, where the disorder degrees of freedom are allowed to reach equilibrium with the rest of the system (Thorpe and Beeman 1976, Falk 1976). Although expected to be quite accurate at high temperatures and small disorder, the annealed approximation is somewhat irrelevant for spin-glass systems: there can be no spin-glass phase, since for spin glasses the disorder certainly has to be frozen on the time scales of experiments. At low temperatures the relaxation of disorder in the annealed approximation builds up strong correlation between the bonds in order to reduce the frustration in the system. For this reason one cannot expect to get meaningful answers

at low temperatures where re-entrance occurs, if at all. In fact, the annealed treatment does give re-entrance for a first-neighbour Ising model with ferromagnetic and antiferromagnetic interactions of different strengths but we believe that this is incorrect for the case of the quenched disorder, as discussed in § 4.

In this paper we study Ising systems whose Hamiltonian can be written

$$H = -\sum_{\langle i,j\rangle} J_{ij} S_i S_j \tag{1}$$

where $S_i = \pm 1$, i and j denote sites on a square or simple cubic lattice and the interaction strengths J_{ij} are random variables. A simple example, which is generally agreed not to have re-entrance, is the first-neighbour model with bonds of equal strength ($\pm J$ model), for which

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J). \tag{2a}$$

As the fraction of ferromagnetic bonds is lowered, the ferromagnetic critical temperature is decreased, and the phase transition is completely destroyed below a critical fraction. Independent of the existence of a SG phase, the FM phase boundary is expected to be vertical (Nishimori 1986, Nemeth 1987). The approximate phase diagram has been obtained in both $d = 2$ and $d = 3$ by high-temperature series expansions (Reger and Zippelius 1985, 1986), as well as by Monte Carlo renormalisation group calculations (Ozeki and Nishimori 1987). Earlier results for the $d = 2$ case were obtained by standard Monte Carlo simulations (Morgenstern and Binder 1980).

A variant of the $\pm J$ model, where the positive and negative bonds have a different magnitude as well as a different fraction, has been investigated by Wolff and Zittartz (1985, 1986). The probability distribution takes the form

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + aJ). \tag{2b}$$

This is one of the models that we will study extensively.

Since the generic re-entrant system, $Eu_xSr_{1-x}S$, has second-neighbour interactions, we also studied a very simple model that mimics many of its properties. A significant difference is that it is an Ising model, whereas the $Eu_xSr_{1-x}S$ system is known to be well modelled by Heisenberg spins (Binder *et al* 1979, Kinzel and Binder 1981). A further difference is the dimensionality of the lattice. Since we were unable to extend our calculations to reasonable system sizes on the three-dimensional FCC lattice, we studied the two-dimensional square lattice with first- and second-neighbour interactions, whose Hamiltonian is written (Binder *et al* 1979, Kinzel and Binder 1981).

$$H = -J_1 \sum_{\langle i,j\rangle} J_{ij}\varepsilon_i\varepsilon_j S_i S_j - J_2 \sum_{\langle\langle k,l\rangle\rangle} \varepsilon_k\varepsilon_l S_k S_l \tag{2c}$$

where $\langle i, j\rangle$ denotes a sum over all first and $\langle\langle k, l\rangle\rangle$ over all second neighbours, $J_2 < 0$ and $\varepsilon_i = 1$ or 0 denotes whether or not a spin is present on site i. As in $Eu_xSr_{1-x}S$ the first-neighbour interactions are purely ferromagnetic ($J_1 > 0$), the second neighbour ones are purely antiferromagnetic ($-J_1/2 \leq J_2 < 0$) and we introduce dilution to model the presence of the randomly placed nonmagnetic atoms: only a fraction x, equal to the average of ε_i, of the spins are present on the lattice. Note that J_1 and J_2 are not random: disorder comes entirely from site dilution. We shall investigate the properties of this model extensively.

3. Techniques

Our goal has been to understand the physical basis of re-entrant spin glasses by studying which short-range models showed this behaviour and which did not. It is not easy,

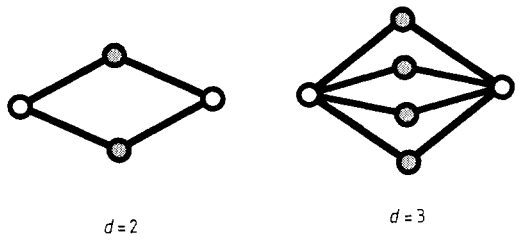

Figure 2. Migdal–Kadanoff decimation clusters in $d = 2$ and $d = 3$. The trace is taken over the shaded spins, and gives a new effective interaction between the unshaded spins.

however, to find a method for short-range systems which is free from crude approximations that might incorrectly *introduce* re-entrance. In random systems it is difficult to use controlled analytical methods, but many numerical techniques, especially simulations, have been successfully used. Standard Monte Carlo simulations are not a suitable technique for the present problem, however, because re-entrant behaviour occurs, if at all, at very low temperatures, where relaxation times are enormously long, so the system does not reach equilibrium in any reasonable time. This is not only because of the low temperatures *per se*, but because the spin-glass phase has in addition anomalously long relaxation times. Consequently, we have chosen two other techniques that are suitable even at low temperatures.

3.1. Migdal–Kadanoff renormalisation

The Migdal–Kadanoff renormalisation-group transformation scheme has proved, despite its simplicity, to be a useful technique for investigations on spin glasses (Young and Stinchcombe 1976, Southern and Young 1977, Kirkpatrick 1977). It did indeed predict correctly that the lower critical dimension d_{lc} in the short-range Ising spin-glass model is below 3. This result is now, after ten years of research, widely accepted (Binder and Young 1986). The method obviously has limitations, and is not expected to give very good estimates of the critical temperatures in spin glasses. It does seem to give, however, the right topology of the phase diagram. Consequently, we decided to examine the low-temperature behaviour of certain Ising spin-glass models to see whether they show re-entrant behaviour. The Migdal–Kadanoff scheme follows the evolution of the bond distribution under the renormalisation group iterations, and locates both the ferromagnetic and the spin-glass phases. Thus it is well suited to examine re-entrant behaviour.

The Migdal–Kadanoff recursion relation for general dimension but fixed rescaling factor $b = 2$ is

$$\beta J'_{ij} = \beta \sum_{l=1}^{2^{d-1}} J'_l = \sum_{l=1}^{2^{d-1}} \tanh^{-1} (\tanh \beta J_{il} \tanh \beta J_{lj}) \tag{3}$$

where β denotes the inverse temperature (measured in units of Boltzmann's constant), d is the dimension of the lattice, the Js represent the bonds, and the primes denote the new values obtained in the current iteration step (Migdal 1976, Kadanoff 1976, Southern and Young 1977). The corresponding clusters for $d = 2$ and $d = 3$ are shown in figure 2. To iterate equation (3) numerically, one constructs a set of N_b bonds drawn independently from the desired initial distribution. One iteration step consists of computing a new set of N_b bonds by selecting 2^d elements of the current set, and combining them according to equation (3) to give one new bond. This step is repeated N_b times to get a

lattice of the original size but with a new distribution of bonds (Southern and Young 1977).

At very low temperatures special care has to be taken with equation (3) since all hyperbolic tangent terms go rapidly to ± 1, so precision is quickly lost. However in this limit equation (3) can be rewritten in the following form

$$J'_{ij} = \sum_{l=1}^{2^{d-1}} J'_l = \sum_{l=1}^{2^{d-1}} \operatorname{sgn}(J_{il}J_{lj}) \min(|J_{il}|, |J_{lj}|) \tag{4}$$

which can be used every time the argument of $\tanh^{-1}$ becomes too close to ± 1.

The various possible phases are located by following the behaviour of the bond distribution under the above iteration. Denoting the mean and the width by J_0 and ΔJ, they evolve as

$J_0 \to 0$	$\Delta J \to 0$	paramagnetic phase	(5*a*)
$J_0 \to \infty$	$\Delta J/J_0 \to 0$	ferromagnetic phase	(5*b*)
$\Delta J \to \infty$	$J_0/\Delta J \to 0$	spin-glass phase	(5*c*)

3.2. Transfer matrix method

Our second and main technique of investigation was the numerical transfer matrix method (Morgenstern and Binder 1980, Morgenstern 1983). It is a numerically exact technique, since it involves the computation of the full trace in the expression for partition function of the system, $Z(T, H)$, for a given configuration of bonds. In particular, it can be used to compute the partition function at very low temperatures, since it does not suffer from long relaxation times, and thus it is ideally suited for our purposes.

In the original form the method was used for the case of small external fields and the quantity of interest was obtained by numerical differentiation of the free energy (Morgenstern and Binder 1980). However, we need two non-vanishing moments of the total magnetisation, $\langle M^2 \rangle$ and $\langle M^4 \rangle$, which cannot be computed by numerical differentiation with sufficient precision. Hence a modification has been implemented, in which the first few terms of the Taylor expansion of $Z(T, H)$ with respect to a uniform external field H, are exactly computed (Saleur and Derrida 1985). They yield directly the cumulants of the magnetisation.

Since all odd terms in H vanish exactly in a finite system, the following Taylor expansion holds for every configuration of bonds:

$$Z = Z(T, H) = Z_0(T) + h^2 Z_2(T) + h^4 Z_4(T) + \ldots \tag{6}$$

where $h = H/T$ and the temperature is measured in units of the Boltzmann constant k_B. The free energy is obtained from

$$F(T, H) = -T \log Z(T, H) \tag{7}$$

and has a Taylor expansion of the form

$$F(T, H) = F_0(T) + \frac{1}{2!} h^2 F_2(T) + \frac{1}{4!} h^4 F_4(T) + \ldots . \tag{8}$$

By expanding equation (7) and comparing the result with equation (8) one finds

$$F_0 = -T \log Z_0 \tag{9a}$$

$$F_2 = -2!\, T \left(\frac{Z_2}{Z_0} \right) \tag{9b}$$

$$F_4 = -4!\,T\left[\frac{Z_4}{Z_0} - \frac{1}{2}\left(\frac{Z_2}{Z_0}\right)^2\right]. \tag{9c}$$

Since the odd moments of the magnetisation vanish (there is no spontaneous magnetisation in a finite system), the first two non-vanishing cumulants are

$$M_2 = \langle M^2\rangle \qquad \text{and} \qquad M_4 = \langle M^4\rangle - 3\langle M^2\rangle^2 \tag{10}$$

which are easily identified as

$$M_2 = -F_2 T \qquad \text{and} \qquad M_4 = -F_4/T. \tag{11}$$

To detect the ferromagnetic phase transition, we use the renormalised coupling constant (Binder 1981):

$$g(T) = \frac{1}{2}\left(3 - \frac{\langle M^4\rangle}{\langle M^2\rangle^2}\right) = -\frac{M_4}{2M_2^2} \tag{12}$$

where the moments and the cumulants are calculated in a system of linear dimension L for a given bond configuration. In the paramagnetic phase ($T > T_c^{FM}$) $g \to 0$ as $N \to \infty$, while in the ferromagnetic phase ($T < T_c^{FM}$) $g \to 1$ as $N \to \infty$, and at the phase transition ($T = T_c^{FM}$) $g = g^*$, *independent* of system size. In other words, $g(T)$ is a particularly convenient quantity to use in a finite size scaling analysis because curves of $g(T)$ against T for different sizes *intersect at the transition temperature*, which is therefore rather easy to locate. In a random system the free energy is obtained by averaging over different configurations of the randomness, i.e. the cumulants of the magnetisation are averaged (cf equations (8) and (11)). Hence it is meaningful to take $g(T)$ as defined by the averaged cumulants, and locate the phase transition by finding g^* from computing $g(T)$ for different system sizes from

$$g(T) = -\frac{[M_4]_{av}}{2[M_2]_{av}^2} \tag{13}$$

where $[\ldots]_{av}$ denotes the configurational average.

The transfer matrix technique for a two-dimensional model restricted to first-neighbour interactions is very straightforward (Morgenstern and Binder 1980, Morgenstern 1983). The first step is to write the partition function of the system as a product of transfer matrices which connect the rows of the system and also contain the interactions within the rows. Consider a system of L rows by M columns, having periodic boundary conditions in the horizontal direction, and free boundary conditions in the vertical direction. The basic idea is to sum over the spins a row at a time. Firstly all the states of the first row are generated and their Boltzmann weights stored in a state vector of length 2^L. Then the first spin of the second row, S_{21}, is included. This would increase the number of weights to 2^{L+1}, were it not for the fact that at this point all spins connected to S_{11} by a bond have already been included. Thus the sum on this variable in $Z(T)$ can be performed at once, keeping the number of weights to 2^L, but having replaced S_{11} by S_{21}. This summation involves the recalculation of all the weights by adding up the contributions to them from the states with $S_{11} = -1$ and $S_{11} = +1$.

One then proceeds by including S_{22} and summing out S_{12}. This is repeated from spin to spin and row to row. There is no need to store more than 2^L weights at any one time, so storage requirements limit this method to $L \leqslant 20$ on today's computers. There is no limitation on the number of rows, however. Instead of the $O(2^{L\times M})$ operations to evaluate the trace in the expression for $Z(T, H)$, and $O(L \times M \times 2^L)$ operations have to be performed. This reduction is what makes the method computationally feasible.

For random systems, where a configurational average over many samples has to be taken, it is the computer time, and not the storage requirement, that limits the practical system size.

The above procedure has to be modified if the model contains second-neighbour as well as first-neighbour interactions. On the square lattice the first spin in the first row S_{11} is connected to the second and last spins, S_{22} and S_{2L}, on the next row, as well as to S_{21}. Therefore we expand the state vector from the states of $(S_{11}S_{12}\cdots S_{1L})$ to those of $(S_{2L}S_{21}S_{11}S_{12}\cdots S_{1L})$, by including the 'ghost' indices S_{2L} and S_{21}. We then work with the 2^{L+2} weights, which specify the possible states of the spins in the first row *and* the first and the last spins in the second row. Just as before, we can now successively sum out and replace the spins of the first row since at any one point all spins connected to the one being summed out are already included. In other words, starting from the configuration $(S_{2L}S_{21}S_{11}S_{12}\cdots S_{1L})$ we replace S_{11} by S_{22} to get $(S_{2L}S_{21}S_{22}S_{12}\cdots S_{1L})$. Then we replace S_{12} by S_{23} and continue in this way until the state is specified by $(S_{2L}S_{21}S_{22}\cdots S_{2L}S_{21})$. At this point all spins of the first row have been summed out. The spins S_{2L} and S_{21} appear twice as indices ('ghosts'). To obtain the state of the second row we have to eliminate the 'ghost' states by retaining only those weights for which the two occurrences of the spins S_{2L} and S_{21} as indices have the same value. This reduces the number of weights by a factor of four and we obtain 2^L weights again, specified by $(S_{21}S_{22}\cdots S_{2L})$.

This extension slows down the computation by about a factor of five, as well as increasing the storage needed by a factor of four. It can still be vectorised using gather/scatter routines (e.g. on a Cray X-MP), and we found that we could collect good statistics on system sizes up to 12×12 within reasonable computer time.

4. Results

4.1. Migdal–Kadanoff renormalisation

The re-entrant phase is easily detected upon convergence of the iteration scheme of equation (3) according to the classification of equation (5). We took typically 10^4 bonds in a set and iterated until convergence occurred to one of the three possible phases. The phase diagram of the two-dimensional random bond Ising model with first-neighbour interactions and probability distribution given by equation (2*b*) is shown in figure 3. The ferromagnetic transition temperature is plotted against the fraction of the ferromagnetic bonds for different values of the strength of the negative bonds a. The phase diagram *does not show re-entrance* for any value of a. The phase boundary seems to be vertical at low temperatures, and there is a strong dependence of $p_c^{FM}(T=0)$ on a. The small deviations from verticality at low temperatures are due to statistical errors in the method. We also carried out the same renormalisation at $T=0$ exactly, using equation (4), since it is conceivable that the re-entrant behaviour occurs in this model at very low temperatures and rather abruptly. Figure 4 shows a plot of the critical curve in the p–a plane at $T=0$, which is the $T\rightarrow 0$ limit of the finite temperature plot in figure 3. It confirms the behaviour we expect from figure 3, i.e. p_c^{FM} $(T=0)$ has strong dependence on a. We shall discuss this in some detail in § 5.

As discussed in § 3, it is trivial to extend these calculations to $d=3$. We show the results in figure 5. As expected (Southern and Young 1977) we find three distinct phases:

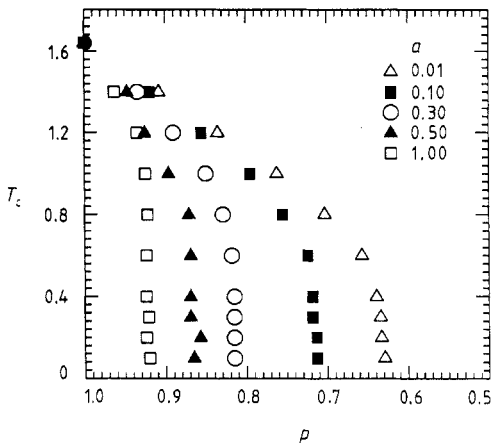

Figure 3. Migdal–Kadanoff phase diagram in $d = 2$ for $T > 0$ of the first-neighbour model with bond probability distribution given by equation (2*b*). The transition temperature between the ferromagnetic and paramagnetic phases is plotted against the fraction of the antiferromagnetic bonds for different values of a, the antiferromagnetic bond strength.

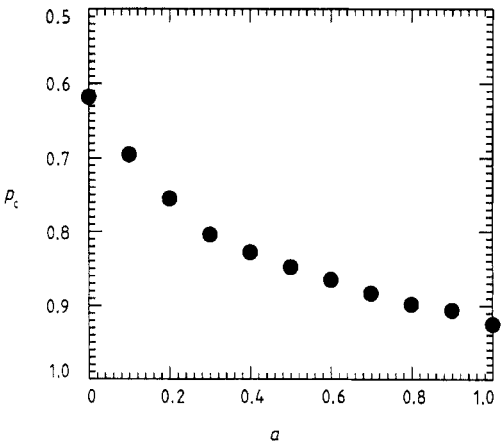

Figure 4. Migdal–Kadanoff phase diagram as in figure 3, but at $T = 0$. The critical fraction separating the ferromagnetic and paramagnetic phases is plotted against the antiferromagnetic bond strength.

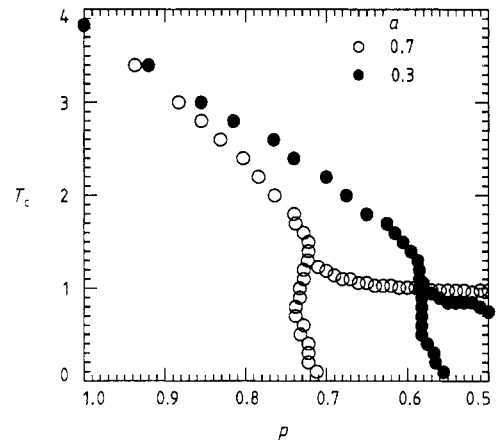

Figure 5. Migdal–Kadanoff phase diagram as in figure 3, but in $d = 3$. The transition temperatures between the ferromagnetic (lower left), paramagnetic (upper right) and spin-glass phases (lower right) are plotted against the fraction of the antiferromagnetic bonds for different values of a, the antiferromagnetic bond strength.

paramagnetic and spin glass. While there is some suggestion of a very small region of re-entrance in the $a = 0.7$ results, this is hardly outside the statistical errors. In any case, we certainly do not obtain re-entrance for the rather large range of concentration observed experimentally.

4.2. Transfer matrix method

The critical temperature of the re-entrant phase is detected by plotting $g(T)$ against T for different linear system sizes L at a given concentration. Suppose we follow the behaviour with decreasing temperature at a given concentration. Just as two such curves intersect at the PM → FM transition, see the discussion after equation (12) above, they should also intersect at the re-entrant FM → PM (or SG, if it exists) transition. Between

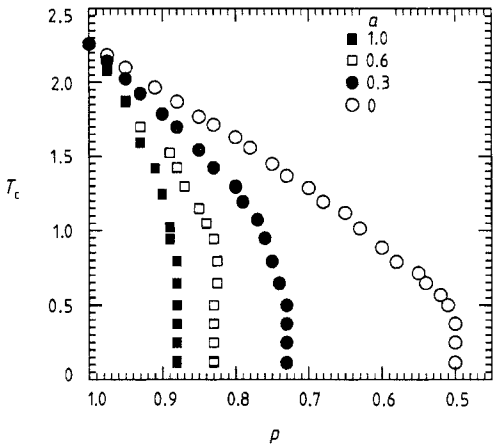

Figure 6. Transfer matrix phase diagram of the same model as in figure 3. The transition temperature is plotted against the fraction of the antiferromagnetic bonds for different values of a, the antiferromagnetic bond strength.

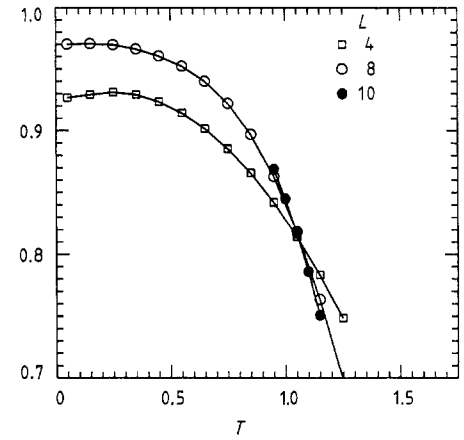

Figure 7. A representative plot of the renormalised coupling constant g against temperature for different linear sizes of the diluted second-neighbour model. The common intersection of the curves is used to locate the critical temperature T_c. The error bars are smaller than the plotted symbols. $J_2 = -0.1, p = 0.8$.

these two critical temperatures $g(T, L_2)$ would have a higher value than $g(T, L_1)$ for $L_2 > L_1$. This behaviour is independent of whether the re-entrant phase is paramagnetic or spin glass, it signals only the *disappearance* of the long-range ferromagnetic order. To save computer time we first bracketed the transition temperature by studying smaller system sizes, and then zeroed in by computing T_c from the largest sizes (10×10 and 12×12) in the bracketed region.

Figure 6 shows the phase diagram for the first-neighbour random bond model using the distribution of equation (2*b*). We plot the ferromagnetic transition temperature as a function of the fraction of the ferromagnetic bonds p for different values of the strength of the antiferromagnetic bonds a. Clearly we find no re-entrance in this model. We also observe a marked dependence of $p_c^{FM}(T = 0)$ on a, in agreement with the above results from Migdal–Kadanoff renormalisation. We have also studied this model for other values of a, in addition to those in figure 6, and found qualitatively the same behaviour in all cases.

Next we discuss the two-dimensional model for $Eu_xSr_{1-x}S$, described by equation (2*c*). The Migdal–Kadanoff method is limited to nearest-neighbour interactions and so cannot be applied here, but, as discussed in § 3.2 above, the transfer matrix approach can be used. For $x = 1.0$ (no dilution) this system has a ferromagnetic ground state for $J_2 = 0$ and a layered antiferromagnetic ground state for $J_2 = -J_1/2$. For $-J_1/2 \leqslant J_2 < 0$ and small enough dilution there is still ferromagnetic order. As described above, we estimate T_c from the intersection of curves for g, and figure 7 shows some typical data. Figure 8 plots the ferromagnetic critical temperature as a function of dilution, $1 - x$, for different ratios of the first and second neighbour interaction. In all cases, there is a concentration range where ferromagnetic order develops upon cooling. However, once developed, this long range order is never lost on further lowering the temperature. Again, we find that p_c^{FM} $(T = 0)$ depends smoothly on the ratio of the interactions $R = J_2/J_1$. An additional test for re-entrance is to look if m_2 decreases with decreasing

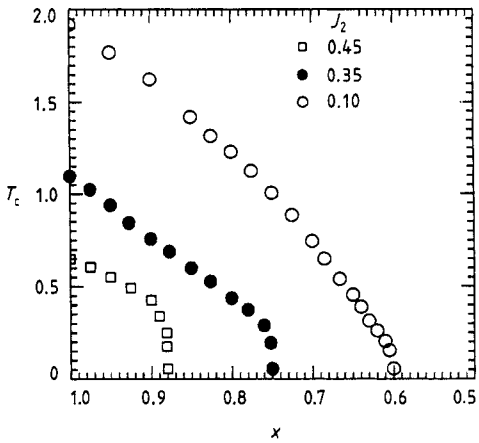

Figure 8. Transfer matrix phase diagram in $d = 2$ of the diluted second-neighbour model described by the Hamiltonian in equation (2*c*). The transition temperature between the ferromagnetic and paramagnetic phases is plotted against the fraction of the non-diluted sites for several values of J_2, the second-neighbour bond strength.

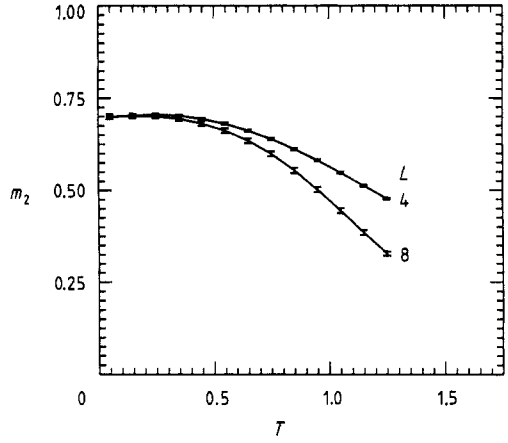

Figure 9. A plot of the second moment of the magnetisation against temperature, for two different linear sizes of the same model as in figure 7. The monotonic behaviour of the curves provides an additional indication of the lack of re-entrance. $J_2 = -0.1, p = 0.4$.

temperature. We found that this never occurred to a significant extent and show some representative data in figure 9.

Although not motivated by any real system, we also did some calculations on another second-neighbour model on the square lattice. The first-neighbour interactions were random and had the same distribution as in equation (2*b*), i.e. a mixture of ferromagnetic and antiferromagnetic bonds of smaller strength, while the second-neighbour couplings were non-random and antiferromagnetic but smaller in magnitude than any one of the first-neighbour interactions. The effect of the antiferromagnetic second-neighbour interactions clearly decreases the ferromagnetic transition temperature for all values of p, and reduces the concentration range where long range ferromagnetic order occurs. We found, however, a phase diagram which is qualitatively similar to that of the other models above. In particular there is no re-entrance. For this reason we do not include any data on this model.

5. Discussion

Our results suggest that re-entrance is *not* a general characteristic of disordered systems with competing interactions, even if the competition is on different length scales, since we find *no re-entrance* in the Ising systems studied. The critical fraction/concentration below which no ferromagnetic order develops was found to depend continuously on the ratio of interaction strengths. We now discuss some recent papers which come to different conclusions.

Wolff and Zittartz have argued (Wolff and Zittartz 1985, 1986) that re-entrance *is* a general feature of disordered models with competing interactions. Their model of Ising spins is described by the Hamiltonian in equation (1) and contains only first-neighbour interactions, drawn from the distribution of equation (2*b*). They argue that the free energy of the system is a function of p, $\tanh J$ and $\tanh aJ$ only. The phase transition is

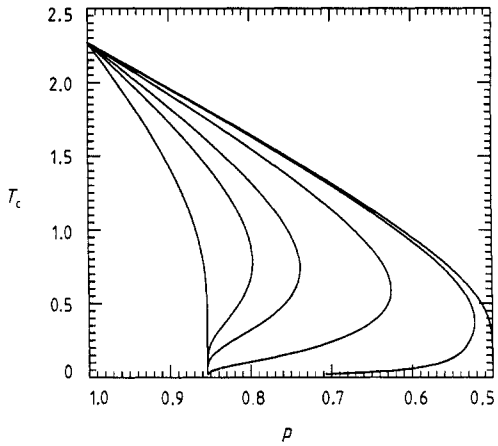

Figure 10. Phase diagram as in figure 3, but in the annealed approximation. The transition temperature between the ferromagnetic and paramagnetic phases is plotted against the fraction of the antiferromagnetic bonds for, from left to right, $a = 1.0, 0.5, 0.3, 0.1, 0.01$ and 0.0, where a is the value of the antiferromagnetic bond strength.

therefore determined from some relation between these quantities, i.e.

$$f(\tanh(J/T), \tanh(aJ/T), p)|_{T=T_c(a,p)} = 0. \tag{14}$$

Assuming that f is an analytic function of its arguments, Wolff and Zittartz argue that the critical fraction, p_1, where $T_c = 0$ is independent of a since, at $T = 0$, one has $\tanh J = -\tanh aJ = 1$ for all values of $-1 \leq a < 0$. For $a = 0$ the bond percolation problem is recovered, and one has a distinct percolation threshold p_0. Since there is a continuous dependence of T_c on a at $T > 0$, it follows that, for a small and negative, the phase boundary will closely follow that of the percolation problem until very low temperatures where it has to bend back to the fraction p_1 of the $\pm J$ model, thereby giving a re-entrant phase boundary.

We believe that this argument is incorrect because the assumption of an analytic function f in equation (14) appears to be true only when there is no frustration. The crucial element of the argument, the existence of the unique critical fraction p_1, resembles very much the annealed approximation (cf figure 10) which does not treat frustration properly either. In fact, it has been shown that improving the annealed approximation, by including frustration to a certain extent, introduces more and more critical fraction values $p_2, p_3, \cdots$ (Georges *et al* 1986, Georges and Le Doussal 1987), and that ultimately all rational values between p_0 and p_1 will occur. Similar results have been obtained within the framework of an effective-field theory, which finds three critical fractions in the parameter range $-1 \leq a \leq 0$ (Sarmento *et al* 1987).

Other arguments can be given that the function f is non-analytic as $T \to 0$. For example, the free energy, when expanded in powers of the hyperbolic tangents, is certainly singular as these tend to plus or minus unity, otherwise the ground-state energy would be incorrectly given as that of the corresponding unfrustrated system. In addition, the Migdal–Kadanoff recursion can be written in terms of hyperbolic tangents, as in equation (3), so one might naively expect that the new distribution would be independent of a at zero temperature. This is not so, however, because the recursion relation *is* singular in this limit, and the interactions are actually given in equation (4).

Several calculations have been performed which find re-entrance using a Bethe–Peierls type approximations (Katsura and Matsuno 1983, Benayad *et al* 1987). This approach, like the annealed approximation and the Wolff–Zittartz argument, does not treat frustration adequately because there are no closed loops on the Bethe lattice.

Indeed, re-entrance appears in the same way in these treatments, namely the critical concentration at zero temperature is found to be constant for a range of parameters in the model. This result is unphysical and disagrees with the results of our transfer matrix calculations.

6. Conclusions

We have presented results of calculations using the numerical transfer matrix technique and the Migdal–Kadanoff renormalisation group scheme applied to the two-dimensional nearest-neighbour random bond and next-nearest-neighbour random site Ising models. The Migdal–Kadanoff calculations were restricted to the nearest-neighbour model, but the technique is sufficiently simple that we were able to investigate the three-dimensional case as well. We searched for re-entrance phenomena in different fraction/concentration ranges of the impurity bonds/sites. In contrast to numerous claims in the recent literature, we have found that there is no re-entrance in these models or, at best, it occurs over a very narrow range of parameters.

Re-entrance is widely observed in experiments, but most experimental systems are Heisenberg-like rather than Ising-like. Our results *do* agree with experiments on the only *Ising* system which has been carefully studied, where long-range order does *not* disappear as the temperature is lowered (Wong *et al* 1985a, b, 1987). Our results are for short-range models and so do not by themselves rule out the possibility, which is perhaps suggested by the work of Viana and Bray (1985), that *long-range* forces may give rise to re-entrance in Ising systems. The results of Wong *et al* (1985a, b, 1987) make this unlikely, however.

It seems possible then that some feature of Heisenberg systems is responsible for re-entrance. We can think of two possibilities: the first is that re-entrance occurs for an isotropic Heisenberg spin glass due to the not yet understood behaviour of the transverse fluctuations. The second is that one needs a coupling between the longitudinal and transverse degrees of freedom, which can arise, for example, from dipole–dipole interactions. We take the results of Borgermann *et al* (1977), which show no re-entrant behaviour in $Eu_xSr_{1-x}Te$, a Heisenberg antiferromagnet, as a suggestion that dipole–dipole interactions may indeed play an important role in re-entrance. Their effect is expected to be greater on ferromagnetic systems than on antiferromagnetic ones because the Fourier transform of the interaction is singular at $k = 0$, so this may explain the lack of re-entrance in Heisenberg antiferromagnets in contrast to Heisenberg ferromagnets.

Although our results have been negative so far, we feel they are valuable because it is important to understand what features of the random models do or do not give re-entrant behaviour. From our results, we believe that disorder and competition are not enough to obtain re-entrant behaviour, which is therefore *not* a general characteristic of all spin-glass systems. The origin of observed re-entrant behaviour in spin glasses is therefore still unclear.

Acknowledgments

We should like to thank W F Wolff for interesting discussions, and A Georges and P Le Doussal for stimulating discussions and correspondence. This work was supported by the National Science Foundation through grants DMR 84-19536 and DMR 85-10593, and through a grant of computer time on the Cray X-MP at the NSF Supercomputer Center in San Diego.

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